Fractional spin through quatum (super)Virasoro algebras.

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Abstract

The splitting of a Q-deformed boson, in the $Q \to q = e^{\frac{2\pi i}{k}}$ limit, is discussed. The equivalence between a Q-fermion and an ordinary one is established. The properties of the quantum (super)Virasoro algebras when their deformation parameter Q goes to a root of unity, are investigated. These properties are shown to be related to fractional supersymmetry and k-fermionic spin.

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1 Introduction

Recently, quantum groups and quantum algebras [1, 2] have played a central role in the development of various arena of mathematics and theoretical physics. Indeed, this new mathematical structure emerge in several context: scattering method [3] as well as Yang-Baxter equations in rational conformal field theory [4, 5, 6, 7]. The representation theory of quantum (super)algebras has been also an object of intensive studies. A vailable are the results for the oscillator representation of quantum (super) algebras. The latters are obtained through consistent realization involving deformed Bose and Fermi operators [8, 9].

In connection with quantum group theory, a new interpretation of fractional supersymetry has been developed in references [10, 11, 12, 13, 14]. In these works, the authors proved that the one-dimensional superspace is isomorphic to the braided line when the deformation parameter is a root of unity. In this case the fractional supersymetry is identified as translational invariance along this line. It is remarkable that at the limit $q = e^{\frac{2i\pi}{k}}$, the braided line [15, 16, 17] is generated by a generalized Grassman variable and an ordinary even one.

Following the same technique, it is proved that internal spin arises naturally at a certain limit of the finite the Q-deformed algebra $U_Q(sl(2))$ [18]. In deed, using Q-Schwinger realization, it is shown that the $U_Q(sl(2))$ is equivalent to a direct product of the finite undeformed algebra $U(sl(2))$ and the deformed one $U_q(sl(2))$ (note that $U_Q(sl(2)) = U_q(sl(2))$ at $Q = q$). Since there exist Q-oscillator realization of all deformed enveloping algebras $U_Q(g)$, it is reasonable to expect these to admit analogous decompositions (or splitting) when $Q \to q$.

In recent years, much interest has been made in the study of the infinite dimensional algebras. The latters appeared in string theories and the two dimensional conformal field theory. These symmetries have been shown to be related to the Virasoro algebra. the two dimensional superconformal field theory can be treated from a group-theoretical point of view, the basic ingredient is the two-dimensional superconformal algebra which is infinite dimensional and called the superVirasoro algebra, see [19] and the references therin. The central extension of superVirasoro leads to so called Ramond-

Neveu-Schwarz algebras [20, 21]. A supersymmetric extension of the work of Curthright and Zachos is given in [22] by using a quantum superspace approach [23, 24].

In the same spirit of quantum (affine) algebras [25, 26], it is more interesting to study the splitting of deformed (super)Virasoro algerbas at a root of unity in order to describe the (super)conformal symmetries at this limit, this is the aim of our present paper.

In this work, we investigate the property of splitting for the deformed (super)Virasoro algebras in the $Q \to q$ limit . At the first steps we start in section (2) by defining the k-fermionic algbras. In section (3) we recall some preliminaries results concerning the property of Q-boson decomposition in the $Q \to q$ limit. We shall first discuss the way in which one obtains two independent objects (an ordinary boson and a k-fermion) from one Q-deformed boson when Q goes to a root of unity. We also show the equivalence between a Q- deformed fermion and a conventional (ordinary or undeformed) fermion. Using the Q-Schwinger realization which presents an interesting properties in the $Q \to q$ limit, we have analyzed the decomposition for the Q-deformed Virasoro algebra in section (4), and the quantum superVirasoro algebra in section (5).

2 Preliminaries about k-fermionic algebra.

The q-deformed bosonic (k-fermion) algebra Σ_q generated by A^+, A^- and number operator N is given by:

$$A^- A^+ - q A^+ A^- = q^{-N} \tag{1}$$

$$A^- A^+ - q^{-1} A^+ A^- = q^{N} \tag{2}$$

$$q^N A^{\pm} q^{-N} = q^{\pm 1} A^{\pm} \tag{3}$$

$$q^N q^{-N} = q^{-N} q^N = 1, \tag{4}$$

where the deformation parameter:

$$q = e^{\frac{2i\pi}{l}},\ l \in N - \{0, 1\}, \tag{5}$$

is a root of unity.

The annihilation operator A^- is hermitian conjugate to creation operator A^+ and N is hermitian also. From equations (1) – (4), it is easy to have the following relations:

$$A^-(A^+)^n = [[n]]q^{-N}(A^+)^{n-1} + q^n(A^+)^n A^- \tag{6}$$

$$(A^-)^n A^+ = [[n]](A^-)^{n-1}q^{-N} + q^n A^+(A^-)^n, \tag{7}$$

where the notation [[ ]] is defined by:

$$[[n]] = \frac{1-q^{2n}}{1-q^2} \tag{8}$$

We introduce a new variable k defined by:

$$k = l \ \text{ for odd values of } \ l, \tag{9}$$

$$k \ = \ \frac{l}{2} \ \text{ for even values of } \ l, \tag{10}$$

such that for odd l (resp. even l), we have $q^k = 1$ (resp. $q^k = -1$). In the particular case $n = k$, equations (6) – (7) permit us to have:

$$A^-(A^+)^k = \pm(A^+)^k A^- \tag{11}$$

$$(A^-)^k A^+ = \pm A^+(A^-)^k, \tag{12}$$

and the equations (1) – (5) yield to:

$$q^N(A^+)^k = (A^+)^k q^N \tag{13}$$

$$q^N(A^-)^k = (A^-)^k q^N \tag{14}$$

One can show that the elements $(A^-)^k$ and $(A^+)^k$ are the elements of the centre of $\sum_q$ algebra (odd values for l); and the irreducible representations are k-dimensional. These two properties lead to:

$$(A^+)^k = \alpha I \tag{15}$$

$$(A^-)^k = \beta I. \tag{16}$$

The extra possibilities parameterized by:

$$(1)\ \ \alpha\ =\ 0,\ \ \beta \neq 0$$

$$(2)\ \ \alpha \neq 0,\ \ \beta = 0$$

$$(3)\ \ \alpha \neq 0,\ \ \beta \neq 0,$$

are not relevant for the considerations of this paper. In the two cases (1) and (2) we have the so-called semi-periodic (semi-cyclic) representation and the case (3) correspond to the periodic one. In what follows, we are interested in a representation of the algebra $\sum_q$ such that the following:

$$(A^{\mp})^k = 0,$$

is satisfied. We note that the algebra $\sum_{-1}$ obtained for $k = 2$, correspond to ordinary fermion operators with $(A^+)^2 = 0$ and $(A^-)^2 = 0$ which reflects the exclusion's Pauli principle. In the limit case where $k \to \infty$, the algebra $\sum_1$ correspond to the ordinary bosons. For other values of k, the k-fermions operators interpolate between fermions and bosons, these are also called anyons with fractional spin in the sense of Majid [15, 16, 17].

3 Fractional spin through Q-boson.

In the previous section, we have worked with q at root of unity. In this case, quantum oscillator (k-fermionic) algebra exhibit a rich representation with very special properties different from the case where q is generic. So, in the first case the Hilbert space is finite dimensional. In contrast, where q is generic, the Fock space is infinite dimensional. In order to investigate the decomposition of Q-deformed boson in the limit $Q \to e^{\frac{2i\pi}{k}}$ we start by recalling the Q-deformed algebra Δ_Q.

The algebra Δ_Q generated by an annihilation operator B^-, a creation operator B^+ and a number operator N_B:

$$B^- B^+ - Q B^+ B^- = Q^{-N_B} \tag{17}$$

$$B^- B^+ - Q^{-1} B^+ B^- = Q^{N_B} \tag{18}$$

$$Q^{N_B} B^+ Q^{-N_B} = Q B^+ \tag{19}$$

$$Q^{N_B} B^- Q^{-N_B} = Q^{-1} B^- \tag{20}$$

$$Q^{N_B} Q^{-N_B} = Q^{-N_B} Q^{+N_B} = 1. \tag{21}$$

From the above equations, we obtain:

$$[Q^{-N_B} B^-, [Q^{-N_B} B^-, [....[Q^{-N_B} B^-, (B^+)^k]_{Q^{2k}} ...]_{Q^4}]_{Q^2}] = Q^{\frac{k(k-1)}{2}} [k]! \tag{22}$$

where the Q-deformed factorial is given by:

$$[k]! = [k][k-1][k-2]..............[1], \tag{23}$$

and:

$$[0]! = 1$$

$$[k] = \frac{Q^k - Q^{-k}}{Q - Q^{-1}} \ .$$

The Q-commutator, in equation (22), of two operators A and B is defined by:

$$[A, B]_Q = AB - QBA$$

The aim of this section is to determine the limit of Δ_Q algebra when Q goes to the root of unity q. The starting point is the limit $Q \to q$ of the equation (22),

$$\lim_{Q \to q} \frac{1}{k} Q^{-N_B} [Q^{-N_B} B^-, [Q^{-N_B} B^-, [....[Q^{-N_B} B^-, (B^+)^k]_{Q^{2k}} ...]_{Q^4}]_{Q^2}]$$

$$= \lim_{Q\to q} \frac{Q^{\frac{k(k-1)}{2}}}{[k]!}[Q^{-N_B}(B^-)^k, (B^+)^k] = q^{\frac{k(k-1)}{2}} \tag{24}$$

This equation can be reduced to:

$$\lim_{Q\to q}[\frac{Q^{\frac{kN_B}{2}}(B^-)^k}{([k]!)^{\frac{1}{2}}}, \frac{(B^+)^k Q^{\frac{kN_B}{2}}}{([k]!)^{\frac{1}{2}}}] = 1. \tag{25}$$

Since q is a root of unity, it is possible to change the sign on the exponent of $q^{\frac{kN_B}{2}}$ terms in the above equation.

We define the operators as in [18]:

$$b^- = \lim_{Q\to q} \frac{Q^{\pm\frac{kN_B}{2}}}{([k]!)^{\frac{1}{2}}}(B^-)^k,\ b^+ = \lim_{Q\to q} \frac{(B^+)^k Q^{\pm\frac{kN_B}{2}}}{([k]!)^{\frac{1}{2}}}, \tag{26}$$

which lead to an ordinary boson algebra noted Δ_0, generated by:

$$[b^-, b^+] = 1. \tag{27}$$

The number operator of this new bosonic algebra defined as the usual case, $N_b = b^+b^-$. At this stage we are in a position to discuss the splitting of Q-deformed boson in the limit $Q \to q$. Let us introduce the new set of generators given by:

$$A^- = B^- q^{-\frac{kN_b}{2}} \tag{28}$$

$$A^+ = B^+ q^{-\frac{kN_b}{2}} \tag{29}$$

$$N_A = N_B - kN_b, \tag{30}$$

which define a k-fermionic algebra:

$$[A^+, A^-]_{q^{-1}} = q^{N_A} \tag{31}$$

$$[A^-, A^+]_q = q^{-N_A} \tag{32}$$

$$[N_A, A^\pm] = \pm A^\pm. \tag{33}$$

It is easy to verify that the two algebras generated by the set of operators $\{b^+, b^-, N_b\}$ and $\{A^+, A^-, N_A\}$ are mutually commutative. We conclude that in the limit $Q \to q$, the Q-deformed bosonic algebra oscillator decomposes into two independent oscillators, an ordinary boson and k-fermion; formally one can write:

$$\lim_{Q \to q} \Delta_Q \equiv \Delta_0 \otimes \Sigma_q,$$

where Δ_0 is the classical bosonic algebra generated by the operators $\{b^+, b^-, N_b\}$.

Similarly, we want to study the Q-fermion algebra at root of unity. To do this, we start by considering the Q- deformed fermionic algebra, noted Ξ_Q:

$$F^- F^+ + Q F^+ F^- = Q^{N_F} \tag{34}$$

$$F^- F^+ + Q^{-1} F^+ F^- = Q^{-N_F} \tag{35}$$

$$Q^{N_F} F^+ Q^{-N_F} = Q F^+ \tag{36}$$

$$Q^{N_F} F^- Q^{-N_F} = Q^{-1} F^- \tag{37}$$

$$Q^{N_F} Q^{-N_F} = Q^{-N_F} Q^{N_F} = 1 \tag{38}$$

$$(F^+)^2 = 0, \;\; (F^-)^2 = 0 \tag{39}$$

We define the new fermionic operators as follow:

$$f^+ = \lim_{Q \to q} F^+ Q^{-\frac{N_F}{2}} \tag{40}$$

$$f^- = \lim_{Q \to q} Q^{-\frac{N_F}{2}} F^-. \tag{41}$$

By a direct calculus, we obtain the following anti-commutation relation:

$$\{f^-, \, f^+\} = 1. \tag{42}$$

Moreover, we have the nilpotency condition:

$$(f^-)^2 = 0, \quad (f^+)^2 = 0. \tag{43}$$

Thus, we see that the Q-deformed fermion reproduce the conventional (ordinary) fermion. The same convention notation permits us to write:

$$\lim_{Q \to q} \Xi_Q \equiv \Sigma_{-1}$$

4 The deformed centerless Virasoro algebra

We apply now the above results to derive the property of decomposition of the quantum Virasoro algebra in the $Q \to q$ limit. Recalling that the classical Virasoro algebra (vir) is generated by the following set of generators $\{l_n, n \in Z\}$ such that:

$$[l_n, l_m] = (m-n) l_{n+m}. \tag{44}$$

It is well known that the algebra (44) can be realized by considering the Schwinger construction. This realization involve one classical (undeformed) bosonic algebra $\{b^+, b^-, N_b\}$ as follows:

$$l_n = (b^+)^{n+1} b^-, \qquad n \geq -1. \tag{45}$$

Recently, a lot of attention has been paid to the Q-deformation of the centreless Virasoro algebra [27, 28, 29, 30] and its central extension [30, 31, 32]. Recalling that the one parameter deformation of the centerless Virasoro algebra (vir_Q) is given by:

$$[L_n, L_m]_{(Q^{m-n}, Q^{n-m})} = [m-n] L_{n+m}, \tag{46}$$

where,

$$[A, B]_{(\alpha, \beta)} = \alpha AB - \beta BA,$$

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

A possible realization of Q-Virasoro generators is given by:

$$L_n = Q^{-N} (B^+)^{n+1} B^-, \tag{47}$$

where B^+ and B^- are Q-deformed creation and annihilation operators respectively, generating the Q-bosonic algebra.

At this stage, our aim is to investigate the limit $Q \rightarrow q$ of the Q-deformed Virasoro algebra vir_Q. As it is already mentioned in the introduction, our analysis is based on the oscillator representation. In the $Q \rightarrow q$ limit, the splitting of Q-deformed boson $\{B^+, B^-,\ N_B\}$ lead to a classical boson $\{b^+, b^-, N_b\}$ given by the equations $(26, 27)$ and a k-fermion algebra $\{A^+, A^-, N_A\}$ given by $eqs(28-30)$. From the undeformed boson, we define the generators j_n:

$$j_n = (b^+)^{n+1} b^-, \qquad n \geq -1, \tag{48}$$

which generates the classical Virasoro algebra:

$$[j_n, j_m] = (m-n) j_{n+m} \tag{49}$$

From the remaining operators $\{A^+, A^-, N_A\}$, one can realize the q-deformed Virasoro algebra vir_q:

$$[J_n, J_m]_{(q^{m-n}, q^{n-m})} = [m-n] J_{n+m} \tag{50}$$

Indeed, the generators defined by:

$$J_n = q^{-N_A} (A^+)^{n+1} A^-, \tag{51}$$

generate the vir_q algebra which is the same version of vir_Q obtained by simply setting $Q = q$, rather than by taking the limit as above.

The generators of classical Virasoro and q-deformed Virasoro algebra are mutually commutative:

$$[J_k, j_l] = 0. \tag{52}$$

So, we obtain the following decomposition:

$$\lim_{Q \rightarrow q} vir_Q = vir_q \otimes vir.$$

We remark that the q-deformed Virasoro algebra exhibit some interesting properties. Indeed, when $m - n = rl$ for any $r \in Z$ the equation (50) is reduced to:

$$[J_n, J_m] = 0. \tag{53}$$

Noticing that, due to the nilpotency condition $(A^+)^l = (A^-)^l = 0$, the generators J_n vanishes for any $n \geq l-1$. This fact constitutes an interesting

property of the deformed Virasoro algebra when the deformation parameter is a root of unity. Namely, for particular value $l = 3$, the q-deformed Virasoro algebra reduces to its subalgebra $su_q(2)$ generated by $\{J_0, J_1, J_{-1}\}$.

5 The Quantum superVirasoro algebra

The classical superVirasoro algebra is generated by the following set of generators $\{L_n, G_n, F_n; n \in Z\}$ satisfying the defining relations [19]:

$$\begin{gathered}
[L_n, L_m] = (n-m)L_{n+m} \\
[F_m, G_n] = G_{n+m} \\
[L_n, F_m] = -mF_{n+m} \\
[F_m, F_n] = 0 \\
[L_m, G_n] = (m-n)G_{n+m} \\
[G_m, G_n] = 0,
\end{gathered} \tag{54}$$

The Z_2 -grading on this superalgebra is defined by requiring that $deg(L_i) = \deg(F_i) = 0$ and $\deg(G_i) = 1$; further, the bracket $[,]$ in relations (54) stands for a graded one:

$$[x, y] = xy - (-1)^{deg(x)deg(y)} yx.$$

The classical superalgebra (54) can be realized by considering the Schwinger construction.

This realization involve two undeformed algebras bosonic $\{b^-, b^+, N_b\}$ and fermionic $\{f^-, f^+, N_f\}$ one:

$$\begin{aligned}
L_n &= -(b^+)^{n+1}b^- \\
G_n &= (b^+)^{n+1}f^+b^- \\
F_n &= (b^+)^n f^+ f^-.
\end{aligned} \tag{55}$$

A one parameter deformation of the superVirasoro algebra is given by [33]:

$$\begin{gathered}
Q^{\mathrm{l-k}} L_l L_k - Q^{\mathrm{k-l}} L_k L_l = [l-k]_Q L_{k+l} \\
F_m G_n - G_n F_m = G_{n+m} \\
L_l F_k - Q^{2k} F_k L_l = -Q[[k]]_Q F_{k+l} \\
Q^{\mathrm{n-m}} F_m F_n - Q^{\mathrm{m-n}} F_n F_m = \lambda[n-m]_Q F_{n+m} \\
Q^{\mathrm{l-k}} L_l G_k - Q^{\mathrm{k-l}} G_k L_l = [l-k]_Q G_{k+l} \\
G_m G_n + G_n G_m = 0,
\end{gathered} \tag{56}$$

where $[x]_Q = \frac{Q^x - Q^{-x}}{Q - Q^{-1}}$, $\quad [[x]]_Q = \frac{1-Q^{2x}}{1-Q^2}$ and $\lambda = Q - Q^{-1}$.

A possible realization of the quantum superVirasoro algebra is given as follows:

$$\begin{aligned} L_n &= -Q^{(1+\frac{n}{2}N)}(B^+)^{n+1}B^- \\ G_n &= Q^{(\frac{n}{2})N}(B^+)^{n+1}f^+B^- \\ F_n &= Q^{(\frac{n}{2})N}(B^+)^{n}f^+f^-, \end{aligned} \tag{57}$$

where B^+ and B^- the Q-deformed bosonic creation and bosonic annihilation operators respectively; $f^+$and f^- are the classical fermionic ones.

Due to the property of Q-boson decomposition in the $Q \to q$ limit, the algbra $\{B^+, B^-, N_B\}$ reproduces an ordinary boson $\{b^+, b^-, N_b\}$ and a k-fermion operator $\{A^+, A^-, N_A\}$. In this limit the Q-fermions become q-fermions which are object equivalent to conventional fermion $\{f^+, f^-, N_f\}$.

The classical superVirassoro algebra $(sVir)$ is given, from the classical boson $\{b^-, b^+, N_b\}$ and the classical fermion $\{f^-, f^+, N_f\}$, by the relations (55).

From the operators $\{A^+, A^-, N_A$, one construct the generators:

$$J_n = q^{-N_A}(A^+)^{n+1}A^-, \tag{58}$$

which generates the q-deformed Virasoro algebra (vir_q) :

$$[J_n, J_m]_{(q^{m-n}, q^{n-m})} = [m-n]_q J_{m+n}. \tag{59}$$

It is to verify that the svir and vir$_q$ are mutually commutative:

$$[J_n, L_n] = [J_n, F_n] = [J_n, G_n] = 0. \tag{60}$$

As a results, we have the following decomposition of the quantum super-Virasoro algebra:

$$\lim_{Q \to q} svir = vir_q \otimes svir.$$

6 Conclusion

We have presented a general method to investigate the $Q \to q = e^{\frac{2\pi i}{k}}$ limit of some Q-deformed infinite algebras based on the decomposition of Q-bosons at this limit. We note that Q-oscillator realization is crucial in this decomposition of these algebras. We have restricted, in this work, our attention

to the quatum Virasoro algebra and quantum superVirasoro algebra. We believe that the techniques and formulas used here will be useful to extend this study to all Q-deformed infinite Lie algebras and superalgebras. This idea will be developed in our forthcoming paper[34].

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